

# GENERALIZED BOUSFIELD LATTICES AND A GENERALIZED RETRACT CONJECTURE

RYO KATO, KATSUMI SHIMOMURA, AND YUTARO TATEHARA

ABSTRACT. In [1], Bousfield studied a lattice (Bousfield lattice) on the stable homotopy category of spectra, and in [5], Hovey and Palmieri made the retract conjecture on the lattice. In this paper we generalize the Bousfield lattice and the retract conjecture to the ones on a monoid. We also determine the structure of typical examples of them, which satisfy the generalized retract conjecture. In particular we give the structure of the Bousfield lattice of the stable homotopy category of harmonic spectra explicitly.

## 1. INTRODUCTION

Let  $\mathcal{M}$  be a closed symmetric monoidal category with zero object, and consider an object  $M$  of it. We call the full subcategory  $\langle M \rangle$  of  $\mathcal{M}$  the *Bousfield class* of  $M$  if it consists of objects  $A$  of  $\mathcal{M}$  such that  $MA = 0$  by its monoidal structure. Then we have a partial order on Bousfield classes by  $\langle M \rangle \leq \langle N \rangle$  if every object of  $\langle N \rangle$  is an object of  $\langle M \rangle$ . Then the subcategories  $\langle S \rangle$  and  $\langle O \rangle$  of the unit  $S$  and the zero  $O$  are the greatest and the least ones in the order, respectively. We call a collection of Bousfield classes a *Bousfield lattice*, and denote it by  $\mathbb{B}(\mathcal{M})$ . In a case where a Bousfield lattice is a set, the partial order introduces a lattice structure to it, and we may investigate it algebraically.

In a sense, the stable homotopy theory is analyzing stable homotopy categories (*cf.* [6]). A stable homotopy category is a symmetric monoidal category, and so we may consider its Bousfield lattice. In particular, T. Ohkawa [8] (*cf.* [2]) showed that the Bousfield lattice  $\mathbb{B}$  of the stable homotopy category of spectra is a set, and then Iyengar and Krause [7] generalized it to a stable homotopy category.

In order to investigate a category, we sometimes classify special subcategories of it. From this viewpoint, we study a Bousfield lattice by classifying localizing subcategories (see [6]). Indeed, every Bousfield class is a localizing subcategory.

In [5], Hovey and Palmieri studied the Bousfield lattice  $\mathbb{B}$  deeply. Furthermore, they proposed many conjectures on the structure of  $\mathbb{B}$ . Among them, there is the retract conjecture, which is one of our main topics. Dwyer and Palmieri [3] constructed a stable homotopy category, where the conjecture does not hold. So far, there seems no nontrivial category in which the conjecture holds. In this paper, we give some examples of categories with the affirmative answer to the conjecture.

As stated above, a Bousfield lattice  $\mathbb{B}(\mathcal{M})$  is a set in some cases. In this case, it is a monoid with multiplication compatible with its order. We introduce the notion of monoidal posets and define a functor  $\beta$  from a subcategory of commutative monoids to the category of monoidal posets in Section two. Then we define a Bousfield lattice of a monoid to be an object in the image of  $\beta$ , which is an analogy of Bousfield

lattices of stable homotopy categories. In particular,  $\mathbb{B}$  has not only a structure of a monoidal poset, but also a Bousfield lattice associated to  $\mathbb{B}$  itself. In section three, we show analogous properties on a Bousfield lattice to those given by Hovey and Palmieri [5] including the following:

**Conjecture 1.1** (Original retract conjecture [5, Conj. 3.12]). *Let  $h$  be the Bousfield class of the mod  $p$  Eilenberg-MacLane spectrum  $H\mathbb{Z}/p$  in the Bousfield lattice  $\mathbb{B}$ . Then, there is a lattice isomorphism  $r_*: \mathbb{B}/J(h) \rightarrow DL$ . Here,  $J(h)$  is an ideal related to  $h$  (see Notation 3.1).*

We generalize it to generalized retract conjectures on a monoidally distributive poset (Conjectures 3.18 and 3.20) and show some facts relating to them. Section four is devoted to determine Bousfield lattices obtained from principal ideal domains, and to show the conjecture true for them. In section five, we study about Bousfield lattices of stable homotopy categories of Bousfield localized spectra, and construct isomorphisms between the Bousfield lattice and a Bousfield lattice given in section four. In particular, we have the following:

**Theorem 1.2.** *The generalized retract conjectures holds on the stable homotopy category of harmonic spectra.*

One of our final goals is to determine the lattice structure of  $\mathbb{B}$ , which seems difficult so much. In the last section, we propose problems on the functor  $\beta$ , whose answers may help us to understand the Bousfield lattice  $\mathbb{B}$ . We expect that these problems give us hints to reach the goal.

## 2. MONOIDAL POSETS AND BOUSFIELD LATTICES

Let  $M$  be commutative monoid with unit 1. We call  $M$  a *monoid with 0* if  $M$  admits an element  $0 \in M$  such that  $0 \cdot x = 0 = x \cdot 0$  for any  $x \in M$ . A typical example of it is a commutative ring ignoring addition. We denote by  $\mathcal{M}_0$  the category consisting of commutative monoids with 0 and monoid homomorphisms preserving zero.

For  $M \in \mathcal{M}_0$ ,  $\beta(M)$  denotes a set consisting of subsets

$$\langle x \rangle = \{y \in M : xy = 0\}$$

of  $M$  for  $x \in M$ .

**Lemma 2.1.**  *$\beta(M)$  for  $M \in \mathcal{M}_0$  is also a monoid with 0 with inherited multiplication. Therefore, we have the canonical epimorphism  $M \rightarrow \beta(M)$  in  $\mathcal{M}_0$ .*

*Proof.* Define a multiplication of  $\beta(M)$  by  $\langle x \rangle \langle y \rangle = \langle xy \rangle$ . We verify it well defined as follows: Assume that  $\langle x_0 \rangle = \langle x_1 \rangle$  and  $\langle y_0 \rangle = \langle y_1 \rangle$ . Then

$$\begin{aligned} zx_0y_0 = 0 &\Leftrightarrow zx_1y_0 = 0 \quad \text{by } \langle x_0 \rangle = \langle x_1 \rangle \\ &\Leftrightarrow zx_1y_1 = 0 \quad \text{by } \langle y_0 \rangle = \langle y_1 \rangle, \end{aligned}$$

and  $\langle x_0y_0 \rangle = \langle x_1y_1 \rangle$ . The elements  $\langle 1 \rangle$  and  $\langle 0 \rangle$  are the unit and the zero elements.  $\square$

*Remark 2.2.* We notice that  $\beta(R) = \mathbb{Z}/2$  if  $R$  is a domain.

**Lemma 2.3.** *Let  $M$  be a monoid with 0. Then  $\beta(M)$  admits a partial order ' $\leq$ ' on  $M$  defined by  $\langle x \rangle \leq \langle y \rangle$  if  $\langle x \rangle \supset \langle y \rangle$ . Besides,  $\langle 1 \rangle$  and  $\langle 0 \rangle$  are the greatest and the least elements, respectively.*

*Proof.* This is trivial since  $\langle 1 \rangle = \{0\}$  and  $\langle 0 \rangle = M$ .  $\square$

By the lemma, a commutative monoid  $\beta(M)$  has also a poset structure. Then we define the following notion by taking their crucial properties.

**Definition 2.4.** A *monoidal poset*  $P = (P, \leq, \cdot, 1, 0)$  is defined by the following data.

- (1)  $(P, \cdot, 1, 0)$  is a monoid with 0.
- (2)  $(P, \leq)$  is a poset.
- (3) The following are equivalent.
  - (a)  $x \leq y$ .
  - (b)  $cy = 0$  for  $c \in P$  implies  $cx = 0$ .

A *monoidal poset map*  $f: P \rightarrow P'$  is an order preserving monoid homomorphism with  $f(0) = 0$ .

Lemma 2.3 implies the following.

**Corollary 2.5.**  $\beta(M)$  for  $M \in \mathcal{M}_0$  is a monoidal poset with  $1 = \langle 1 \rangle$  and  $0 = \langle 0 \rangle$ .

**Lemma 2.6.** Let  $M$  be a monoidal poset. Then,  $\beta(M) = M$  as a monoidal poset.

*Remark 2.7.* A monoidal poset seems a lattice, but unfortunately it is not true. Indeed, we have an example: Consider a monoidal poset  $M = \{1, x_i, y_i, w, 0: i = 1, 2\}$  with multiplication

1	$x_1$	$x_2$	$y_1$	$y_2$	$w$
$x_1$	$w$	$w$	0	$w$	0
$x_2$	$w$	$w$	$w$	0	0
$y_1$	0	$w$	0	0	0
$y_2$	$w$	0	0	0	0
$w$	0	0	0	0	0

Then, the join of  $y_1$  and  $y_2$  does not exist.

Let  $\mathcal{MP}$  denote the category of monoidal posets and monoidal poset maps. Then  $\mathcal{MP} \subset \mathcal{M}_0$ .

**Lemma 2.8.** Let  $M$  be a monoidal poset. Then,  $xz \leq yw$  if  $x \leq y$  and  $z \leq w$ . In particular, if  $x \leq y$ , then  $xz \leq yz$  for any  $z$ .

**Proposition 2.9.** The category  $\mathcal{MP}$  admits a direct product.

*Proof.* Let  $\{M_\lambda\}$  be a family of monoidal posets. Then, we have a direct product  $\prod_\lambda M_\lambda$  of monoids. Consider an order ' $\leq$ ' on  $\prod_\lambda M_\lambda$  defined by  $(x_\lambda) \leq (y_\lambda)$  if  $(c_\lambda)(y_\lambda) = (0)$  implies  $(c_\lambda)(x_\lambda) = (0)$ . It is straightforward to verify this is the desired direct product.  $\square$

**Lemma 2.10.** Let  $\{M_\lambda\}$  be a family of monoidal posets. Then,  $\langle x_\lambda \rangle \leq \langle y_\lambda \rangle$  for all  $\lambda$  if and only if  $\langle (x_\lambda) \rangle \leq \langle (y_\lambda) \rangle$ . Here,  $\langle x_\lambda \rangle, \langle y_\lambda \rangle \in \beta(M_\lambda)$  and  $\langle (x_\lambda) \rangle, \langle (y_\lambda) \rangle \in \beta(\prod_\lambda M_\lambda)$ .

*Proof.* Assume that  $\langle x_\lambda \rangle \leq \langle y_\lambda \rangle$  for any  $\lambda$ . Then

$$\begin{aligned}
(c_\lambda)(y_\lambda) = 0 &\Rightarrow c_\lambda y_\lambda = 0 \text{ for any } \lambda \\
&\Rightarrow c_\lambda x_\lambda = 0 \text{ for any } \lambda (\because \langle x_\lambda \rangle \leq \langle y_\lambda \rangle) \\
&\Rightarrow (c_\lambda)(x_\lambda) = 0,
\end{aligned}$$

Conversely, suppose that  $\langle(x_\mu)\rangle \leq \langle(y_\mu)\rangle$ . Then, for any  $\lambda$ ,

$$\begin{aligned} y_\lambda c_\lambda = 0 &\Rightarrow (y_\lambda)(c_\lambda)_0 = 0 \\ &\Rightarrow (x_\lambda)(c_\lambda)_0 = 0 \quad (\because \langle(x_\mu)\rangle \leq \langle(y_\mu)\rangle) \\ &\Rightarrow x_\lambda c_\lambda = 0 \end{aligned}$$

in  $M_\lambda$ , where  $(c_\lambda)_0$  denotes an element  $(x_\mu)$  such that  $x_\lambda = c_\lambda$  and  $x_\mu = 0$  for  $\mu \neq \lambda$ .  $\square$

**Corollary 2.11.** *Let  $\{M_\lambda\}$  be a family of monoidal posets. Define an order  $\leq'$  on the set  $\prod_\lambda M_\lambda$  by  $(x_\lambda) \leq' (y_\lambda)$  if  $x_\lambda \leq y_\lambda$  for all  $\lambda$ . Then it is equivalent to the order above  $\leq$ .*

**Corollary 2.12.** *Let  $\{M_\lambda\}$  be a family of monoidal posets. Then,  $\bigvee_\mu (x_\lambda^\mu) = (\bigvee_\mu x_\lambda^\mu)$  for any subset  $\{(x_\lambda^\mu)\}_\mu \subset \prod_\lambda M_\lambda$ .*

*Proof.* Since  $(x_\lambda^\mu) \leq (\bigvee_\mu x_\lambda^\mu)$  for all  $\mu$ ,  $\bigvee_\mu (x_\lambda^\mu) \leq (\bigvee_\mu x_\lambda^\mu)$ . If  $(x_\lambda^\mu) \leq (z_\lambda)$ , then  $x_\lambda^\mu \leq z_\lambda$ , and so  $\bigvee_\mu x_\lambda^\mu \leq z_\lambda$ , that is,  $(\bigvee_\mu x_\lambda^\mu) \leq (z_\lambda)$ . Therefore,  $\bigvee_\mu (x_\lambda^\mu) = (\bigvee_\mu x_\lambda^\mu)$  by definition.  $\square$

We call an epimorphism  $f: M \rightarrow N$  of  $\mathcal{M}_0$  *strong* if  $f(x) = 0$  if and only if  $x = 0$ .

We define a map  $\beta(f): \beta(M) \rightarrow \beta(N)$  by assigning  $\langle x \rangle$  to  $\langle f(x) \rangle$ .

**Lemma 2.13.** *For a strong epimorphism  $f: M \rightarrow N$ , the map  $\beta(f)$  is not only a monoidal poset map but also a strong epimorphism.*

*Proof.* Since  $f$  is a strong epimorphism,  $c \cdot f(x) = 0 \Leftrightarrow f(c') \cdot f(x) = 0 \Leftrightarrow f(c' \cdot x) = 0 \Leftrightarrow c' \cdot x = 0$  for an element  $c'$  such that  $f(c') = c$ . This shows that  $\langle x \rangle = \langle y \rangle$  implies  $\langle f(x) \rangle = \langle f(y) \rangle$ . It is easy to see that  $\beta(f)$  is a strong epimorphism.  $\square$

We also consider the subcategories  $\mathcal{M}_0^{epi}$  and  $\mathcal{MP}^{epi}$  of  $\mathcal{M}_0$  and  $\mathcal{MP}$ , respectively, obtained by restricting morphisms to strong epimorphisms.

**Corollary 2.14.** *The operation  $\beta$  above defines a functor  $\beta: \mathcal{M}_0^{epi} \rightarrow \mathcal{MP}^{epi} \subset \mathcal{M}_0^{epi}$ .*

By the above argument, we redefine Bousfield lattices as follows. The definition is one of our main topics in this paper.

**Definition 2.15.** For a monoid  $M \in \mathcal{M}_0^{epi}$  we call a monoidal poset  $\beta(M)$  the *Bousfield lattice associated to  $M$* .

In earlier papers, a Bousfield lattice is made from a closed symmetric monoidal category with a zero object. However, their set theoretic confusion complicates our argument too much. Our new definition settles this problem, and the following proposition says that this argument is consistent.

**Proposition 2.16.** *The Bousfield lattice  $\mathbb{B}$  of the stable homotopy category of spectra is a Bousfield lattice in the sense of our definition.*

*Proof.* By forgetting the ordering on  $\mathbb{B}$ , we regard  $\mathbb{B}$  as a monoid with  $1 = \langle S \rangle$  and  $0 = \langle * \rangle$ . Then it is clear that  $\beta(\mathbb{B}) = \mathbb{B}$ .  $\square$

**Proposition 2.17.** *The functor  $\beta$  satisfies the following:*

- (1)  $\beta(\prod_{\lambda} M_{\lambda}) = \prod_{\lambda} \beta(M_{\lambda})$ .
- (2)  $\beta\beta(M) = \beta(M)$ .

*Proof.* (1) Let  $\{p_{\lambda}: \beta(\prod_{\lambda} M_{\lambda}) \rightarrow \beta(M_{\lambda})\}$  be a family of epimorphisms defined by  $\langle\langle x_{\lambda} \rangle\rangle \mapsto \langle x_{\lambda} \rangle$ , and  $\{f_{\lambda}: W \rightarrow \beta(M_{\lambda})\}$  a family of poset maps. We notice that  $p_{\lambda}$  is well defined by Lemma 2.10. For an element  $w \in W$ , we take an element  $w_{\lambda} \in W_{\lambda}$  so that  $f_{\lambda}(w) = \langle w_{\lambda} \rangle$ , and define  $g: W \rightarrow \beta(\prod_{\lambda} M_{\lambda})$  by  $g(w) = \langle\langle w_{\lambda} \rangle\rangle$ . Then  $g$  is also a well defined poset map by Lemma 2.10 and

$$p_{\lambda}g(w) = p_{\lambda}(\langle\langle w_{\lambda} \rangle\rangle) = \langle w_{\lambda} \rangle = f_{\lambda}(w).$$

Suppose that there is another poset map  $g': W \rightarrow \beta(\prod_{\lambda} M_{\lambda})$  satisfying  $p_{\lambda}g'(w) = f_{\lambda}(w)$  for  $w \in W$ , and  $g'$  assigns  $w$  to  $\langle\langle w'_{\lambda} \rangle\rangle$ . Then

$$\begin{aligned} p_{\lambda}g'(w) = f_{\lambda}(w) \text{ for any } \lambda &\Leftrightarrow \langle w'_{\lambda} \rangle = \langle w_{\lambda} \rangle \text{ for any } \lambda \\ &\Leftrightarrow \langle\langle w'_{\lambda} \rangle\rangle = \langle\langle w_{\lambda} \rangle\rangle \text{ } (\because \text{Lemma 2.10}) \\ &\Leftrightarrow g'(w) = g(w). \end{aligned}$$

Therefore,  $\beta(\prod_{\lambda} M_{\lambda})$  is the product  $\prod_{\lambda} \beta(M_{\lambda})$ .

(2) is seen by Lemma 2.6. □

### 3. RETRACT CONJECTURE

From now on, we assume that every monoidal poset is a complete lattice.

Since a monoidal poset  $M$  is a sup-lattice with the least element  $0 = \langle 0 \rangle$ ,  $M$  is a bounded lattice.

**Notation 3.1.** For a monoidal poset  $M$ , we define the following notations.

$$\begin{aligned} a_M(x) &:= \bigvee \{y \in M : xy = 0\} \text{ for } x \in M, \\ DL(M) &:= \{x \in M : x^2 = x\}, \\ r_M(x) &:= \bigvee \{w \in DL(M) : w \leq x\} \text{ for } x \in M, \\ J_M(x) &:= \{y \in M : y \leq x \wedge a_M(x)\} \text{ for } x \in M, \\ N(M) &:= \{x \in M : x^n = 0 \text{ for some } n \geq 1\}, \\ A(M) &:= \{x \in M : r_M(x) = 0\}. \end{aligned}$$

We will omit  $M$  from notations, if  $M$  is clear from the context.

The subposet  $DL(M)$  is also a complete lattice. Indeed the following holds.

**Proposition 3.2.**  $DL(M)$  is closed under the join  $\vee$ .

*Proof.* By Lemma 2.8,  $(\bigvee_{\lambda \in \Lambda} x_{\lambda})^2 \leq (\bigvee_{\lambda \in \Lambda} x_{\lambda})$ . Suppose that  $x_{\lambda}$  is in  $DL$  for  $\lambda \in \Lambda$ . Then,  $x_{\lambda} = x_{\lambda}^2 \leq (\bigvee_{\lambda \in \Lambda} x_{\lambda})^2$ , and so  $\bigvee_{\lambda \in \Lambda} x_{\lambda} \leq (\bigvee_{\lambda \in \Lambda} x_{\lambda})^2$ . □

**Lemma 3.3.** In  $DL(M)$ , the meet of  $x$  and  $y$  is  $xy$ .

*Proof.* Since  $x \wedge y \leq x$  and  $x \wedge y \leq y$ , if  $x \wedge y \in DL(M)$  then  $x \wedge y \leq xy$ . □

*Remark 3.4.*  $DL(M)$  is not always sublattice of  $M$  by Lemma 3.3.

For investigating the original Bousfield lattice  $\mathbb{B}$ , the operations  $r$  and  $a$  play important roles (see [5]). Hereafter we try to give their properties analogously on good monoidal posets.

**Proposition 3.5.** Let  $M$  be a monoidal poset, and  $r = r_M: M \rightarrow M$  be the map defined in Notation 3.1.

- (1)  $r$  is order-preserving i.e.  $x \leq y$  implies  $r(x) \leq r(y)$ .

- (2)  $r(x)^2 = r(x)$  and  $r^2(x) = r(x)$  for  $x \in M$ .
- (3)  $r(x) \leq x^n$  for any  $n \geq 1$ .
- (4)  $r(xy) = r(x)r(y) = r(x \wedge y)$  for  $x, y \in M$ .

*Proof.* (1) is trivial, and (2) follows from Proposition 3.2. For (3),  $r(x) \leq x$  by definition, and we have  $r(x) = r(x)^n \leq x^n$ .

Since  $r(x)r(y) \leq xy$  and  $r(x)r(y) \in DL(M)$ , we have  $r(x)r(y) \leq r(xy)$ . We also see  $r(x \wedge y) \leq r(x)r(y)$ , since  $r(x \wedge y) \leq r(x)$  and  $r(x \wedge y) \leq r(y)$ . Therefore,  $r(xy) \leq r(x \wedge y) \leq r(x)r(y) \leq r(xy)$ , and obtain (4).  $\square$

The behavior of the map  $r$  is the same as the one on  $\mathbb{B}$ , but not that of the operation  $a$ . Indeed, for any  $x \in M$  and  $\{y_\lambda\}_\lambda \subset M$ , the relation  $x(\bigvee_\lambda y_\lambda) \geq \bigvee_\lambda(xy_\lambda)$  is not always equal. To make the operator  $a$  have a good nature, we introduce a following notion.

**Definition 3.6.** A monoidal poset  $M$  is a *monoidally distributive* poset if  $M$  satisfies that  $x(\bigvee_\lambda y_\lambda) = \bigvee_\lambda(xy_\lambda)$  for any  $x \in M$  and  $\{y_\lambda\}_\lambda \subset M$ .

*Remark 3.7.*  $DL(M)$  is a distributive lattice if  $M$  is a monoidally distributive poset by Lemma 3.3.

In the same way as [5], we have

**Proposition 3.8.** *Let  $M$  be a monoidally distributive poset. Then,*

- (1)  $a(-)$  is order-reversing.
- (2)  $xy = 0$  if and only if  $x \leq a(y)$ .
- (3)  $aa(x) = x$ .

**Lemma 3.9.** *Let  $M$  be a monoidally distributive poset. Fix  $c \in M$  such that  $c^n = 0$  for a positive integer  $n$ . Then, for any  $x \in M$ ,  $(x \vee c)^n \leq x$  and  $r(x \vee c) = r(x)$ .*

*Proof.* Under the assumption, we compute

$$\begin{aligned} (x \vee c)^n &= x^n \vee x^{n-1}c \vee \dots \vee xc^{n-1} \\ &= x(x^{n-1} \vee x^{n-2}c \vee \dots \vee c^{n-1}) \leq x \end{aligned}$$

for any  $x \in M$ . So, if  $z \leq x \vee c$  for  $z \in DL(M)$ , then  $z \leq x$ . Thus,  $r(x \vee c) = r(x)$  by definition of  $r$ .  $\square$

**Proposition 3.10.** *Let  $M$  be a monoidally distributive poset. Then  $J_M(x) \subset N(M) \subset A(M)$  for any  $x \in M$ .*

*Proof.* Since  $(x \wedge a_M(x))(x \wedge a_M(x)) \leq xa_M(x) = 0$  by Proposition 3.8(2), we have  $J_M(x) \subset N(M)$ . Suppose that  $x^n = 0$ , then  $r(x) = r(x)^n = r(x^n) = r(0) = 0$  by Proposition 3.5 (4). So we have  $N(M) \subset A(M)$ .  $\square$

**Proposition 3.11.** *Let  $M_\lambda$  be a monoidal poset for any  $\lambda \in \Lambda$ . Then,*

- (1)  $r((x_\lambda)) = (r(x_\lambda))$  for any  $(x_\lambda) \in \prod_\lambda M_\lambda$ .
- (2)  $r$  preserves arbitrary joins on  $M_\lambda$  for any  $\lambda \in \Lambda$  if and only if  $r$  preserves arbitrary joins on  $\prod_\lambda M_\lambda$ .

*Proof.* (1) is given by corollary 2.12.

(2) Suppose that  $r$  preserves arbitrary joins on  $M_\lambda$  for any  $\lambda \in \Lambda$ . Then, for  $\{(x_\lambda^\mu)\}_\mu \subset \prod_\lambda M_\lambda$ ,

$$\begin{aligned} r(\bigvee_\mu (x_\lambda^\mu)) &= r(\bigvee_\mu x_\lambda^\mu) \quad (\because \text{corollary 2.12}) \\ &= (r(\bigvee_\mu x_\lambda^\mu)) \quad (\because (1)) \\ &= \bigvee_\mu r(x_\lambda^\mu) \\ &= \bigvee_\mu (r(x_\lambda^\mu)) \quad (\because \text{corollary 2.12}) \end{aligned}$$

Therefore,  $r$  preserves arbitrary joins on  $\prod_\lambda M_\lambda$ .

Conversely, under the assumption,

$$\begin{aligned} (r(\bigvee_\mu x_\lambda^\mu)) &= r(\bigvee_\mu x_\lambda^\mu) \quad (\because (1)) \\ &= r(\bigvee_\mu (x_\lambda^\mu)) \quad (\because \text{corollary 2.12}) \\ &= \bigvee_\mu (r(x_\lambda^\mu)) \\ &= \bigvee_\mu r(x_\lambda^\mu) \quad (\because \text{corollary 2.12}) \end{aligned}$$

It follows that  $r$  preserves arbitrary joins on  $M_\lambda$  for any  $\lambda \in \Lambda$  as desired.  $\square$

*Remark 3.12.* We notice that  $M_\lambda$  is a monoidally distributive poset for any  $\lambda \in \Lambda$  if and only if  $\prod_{\lambda \in \Lambda} M_\lambda$  is a monoidally distributive poset. Indeed, if  $M_\lambda$  is a monoidally distributive poset for any  $\lambda \in \Lambda$ , then  $(c_\lambda)(\bigvee_\mu (x_\lambda^\mu)) = (c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (\bigvee_\mu c_\lambda x_\lambda^\mu) = \bigvee_\mu (c_\lambda x_\lambda^\mu)$  for  $(c_\lambda) \in \prod_\lambda M_\lambda$  and  $\{(x_\lambda^\mu)\}_\mu \subset \prod_\lambda M_\lambda$  by corollary 2.12. Thus,  $\prod_\lambda M_\lambda$  is a monoidally distributive poset. Conversely, under the assumption  $(c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (c_\lambda)(\bigvee_\mu (x_\lambda^\mu)) = \bigvee_\mu (c_\lambda x_\lambda^\mu) = (\bigvee_\mu c_\lambda x_\lambda^\mu)$  by corollary 2.12. Therefore,  $M_\lambda$  is a monoidally distributive poset for any  $\lambda \in \Lambda$  by Lemma 2.10.

Recall that an ideal  $I$  of a poset is defined by

- (1) If  $x \in I$ , and  $y \leq x$ , then  $y \in I$ , and
- (2) For  $x, y \in I$ , there is an element  $z \in I$  such that  $x \leq z$  and  $y \leq z$ .

Suppose that a monoidal poset  $M$  is an ordinary lattice. Then, an ideal of  $M$  is also an ideal as a lattice, and for an ideal  $I$ ,  $M/I$  is the lattice of equivalent classes under the equivalent relation defined by

$$(3.13) \quad x \sim y \quad \text{if and only if } x \vee c = y \vee c \text{ for some } c \in I$$

with order given by  $[x] \leq [y] \Leftrightarrow x \vee c \leq y \vee c$  for some  $c \in I$ . We notice that  $M/I$  is complete if  $M$  and  $I$  are complete. If  $M$  is monoidally distributive, then  $M/I$  has the multiplication  $[x][y] := [xy]$ . Indeed, if  $x \vee i = x' \vee i$  and  $y \vee j = y' \vee j$  for  $x, x', y, y' \in M$  and  $i, j \in I$ , then  $(x \vee i)(y \vee j) = (x' \vee i)(y' \vee j)$  turns into

$$\begin{aligned} xy \vee (x \vee i)j \vee (y \vee j)i &= x'y' \vee (x' \vee i)j \vee (y' \vee j)i \\ &= x'y' \vee (x \vee i)j \vee (y \vee j)i. \end{aligned}$$

Since  $(x \vee i)j \vee (y \vee j)i \in I$ , the multiplication is well defined.

*Remark 3.14.*  $M/I$  is not always a monoidal poset. Indeed, we have an example: Let  $M = \{1, x, y, 0\}$  be a monoidal poset with multiplication  $x^2 = x, xy = 0, y^2 = 0$ . Then, for the ideal  $I = \{y, 0\}$ ,  $M/I = \{1, x, 0\}$  and  $\beta(M/I) = \{1, 0\}$ . Since  $M/I \neq \beta(M/I)$ ,  $M/I$  is not a monoidal poset by Lemma 2.6.

**Lemma 3.15.** *Let  $M$  be a monoidally distributive poset. Then,  $N(M)$  is an ideal of  $M$  and  $J_M(x)$  is a principal ideal of  $M$  for any  $x \in M$ .*

*Proof.* Suppose that  $x^n = 0$  and  $y^m = 0$ . Then,  $(x \vee y)^{n+m} = \bigvee_{a+b=n+m} x^a y^b$ . Since if  $a < n$  then  $b \geq m$ ,  $(x \vee y)^{n+m} = 0$ . So  $N(M)$  is an ideal of  $M$ . By definition,  $J_M(x)$  is a principal ideal of  $M$ .  $\square$

Here, consider the following correspondence:

$$r_* : M/I \rightarrow DL(M); [x] \mapsto \{r(y) : y \in [x]\}$$

We notice that  $r_*$  is surjection if  $r_*$  is a mapping.

**Theorem 3.16.** *Let  $M$  be a monoidally distributive poset and  $I$  an ideal in  $M$ .*

- (1) *If  $I$  is contained in  $N$ , then  $r_*$  is a mapping.*
- (2) *If  $r_*$  is a mapping, then  $I \subset A$ .*
- (3) *If  $r_*$  is an injection, then  $I = A$ .*
- (4) *If  $r_*$  is an injection and  $I \subset N$ , then:*
  - (a) *For any  $x$  and  $y$  in  $M$ ,  $r(x \vee y) = r(x) \vee r(y)$  holds. In particular, if  $I$  is a principal ideal, then  $r$  preserves arbitrary joins.*
  - (b) *For any  $x \in M$ , there exists an integer  $n$  such that  $x^n = r(x)$ .*

*Proof.* (1) If  $x \vee c = y \vee c$  for  $x, y \in M$  and  $c \in I \subset N$ , then  $r(x) = r(y)$  by Lemma 3.9.

(2) For  $x \in I$ ,  $[x] = 0 = [0]$  in  $M/I$ , and so  $r(x) = r_*([x]) = r_*([0]) = r(0) = 0$ . Thus,  $x \in A$ .

(3) For  $x \in A$ ,  $r_*([x]) = r(x) = 0 = r_*([0])$ . It follows that  $[x] = [0]$ , since  $r_*$  is an injection, which implies  $x \in I$ . So we obtain  $A = I$  by (2).

(4) For  $x \in M$ ,  $r_*([x]) = r(x) = r^2(x) = r_*([r(x)])$  and  $[x] = [r(x)]$ , since  $r_*$  is an injection. So we have an element  $c_x \in N$  such that  $x \vee c_x = r(x) \vee c_x$ , and then:

- (a) Since  $x \vee y \vee c_x \vee c_y = r(x) \vee r(y) \vee c_x \vee c_y$ ,  $r(x \vee y) = r(x) \vee r(y)$  by Lemma 3.9. Suppose that  $I$  is a principal ideal and take a generator  $m$  of  $I$ . Then,  $(\bigvee_{\lambda} x_{\lambda}) \vee m = (\bigvee_{\lambda} r(x_{\lambda})) \vee m$  for any subset  $\{x_{\lambda}\}_{\lambda} \subset M$ . Therefore  $r(\bigvee_{\lambda \in \Lambda} x_{\lambda}) = \bigvee_{\lambda \in \Lambda} r(x_{\lambda})$  by Lemma 3.9.
- (b) Since there exists an integer  $n$  such that  $c_x^n = 0$ ,

$$x^n \leq (x \vee c_x)^n = (r(x) \vee c_x)^n \leq r(x).$$

by Lemma 3.9.  $\square$

Hovey and Palmieri introduced a map  $r_* : M/J(h) \rightarrow DL$ , and proposed Conjecture 1.1 in the introduction. Here, we generalize the map to our setting.

**Lemma 3.17.** *The map  $r_M : M \rightarrow M$  for a monoidal poset  $M$  factors through  $DL(M)$ . Furthermore, it induces the map  $r_* : M/J_M(y) \rightarrow DL(M)$  for  $y \in M$  assigning the class  $[x]$  to  $r_M(x)$ .*

*Proof.* The former statement follows from Proposition 3.5(2), and the latter from Proposition 3.10 and Proposition 3.16(1).  $\square$

By Theorem 3.16, we see that  $J(h) = A$  if Conjecture 1.1 holds. This makes us conjecture the following:

**Conjecture 3.18** (Generalized retract conjecture 1 (GRC1)). *Let  $M$  be a monoidal poset. If  $M$  is a complete lattice and monoidally distributive, then  $r_* : M/A \rightarrow DL$  is a lattice isomorphism.*

*Remark 3.19.* We notice that  $A(M)$  is not always an ideal of a monoidal poset  $M$ . GRC1 is false unless  $A(M)$  is an ideal.



**Conjecture 3.20** (Generalized retract conjecture 2 (GRC2)). *Let  $M$  be a monoidal poset. If  $M$  is a complete lattice and monoidally distributive, then  $r_*: M/N \rightarrow DL(M)$  is a lattice isomorphism.*

By Theorem 3.16 (3), we see the following:

**Corollary 3.21.** *GRC2 implies GRC1.*

**Example 3.22.** *Consider a monoidal poset  $M = \beta(\mathbb{Z}/2^m\mathbb{Z})$ . Then,*

$$\begin{aligned} M &= \{1, 2, 2^2, \dots, 2^{m-1}, 2^m = 0\}, \\ DL(M) &= \{1, 0\} \text{ and} \\ N(M) &= \{2, 2^2, \dots, 2^{m-1}, 0\}. \end{aligned}$$

And so  $M/N(M) \cong DL(M)$ . That is, GRC2 holds on  $\beta(\mathbb{Z}/2^m\mathbb{Z})$ .

**Theorem 3.23.** *For a monoidally distributive poset  $M$ , the following are equivalent.*

- (1)  $r_*: M/N \rightarrow DL$  is an isomorphism.
- (2) Any class  $[x] \in M/N$  satisfies  $[x^2] = [x]$ .

*Proof.* The statement (1) implies (2), since  $r_*([x]) = r_*([x^2])$ .

For the converse, it suffices to show that  $r_*$  is injective. If  $[x^2] = [x]$ , then  $[x] = [x^n]$  for any  $n > 0$  by induction. So, we have an element  $c_x \in N$  for each  $x \in M$  such that

$$(3.24) \quad x \vee c_x = x^n \vee c_x \text{ for any } n > 0.$$

Since  $c_x \in N$ , we have an integer  $L = L(x) > 0$  such that  $c_x^L = 0$ . Then

$$x^L \leq (x \vee c_x)^L = (x^n \vee c_x)^L \leq x^n$$

for any  $n > 0$  by Lemma 3.9. In particular,  $x^L = (x^L)^2$  and so

$$(3.25) \quad x^{L(x)} = r(x).$$

by Proposition 3.5.

Now suppose that  $r_*([x]) = r_*([y])$ . Then  $r(x) = r(y)$ , and  $x^{L(x)} = y^{L(y)}$  by (3.25). By (3.24),

$$x \vee c_x \vee c_y = x^{L(x)} \vee c_x \vee c_y = y^{L(y)} \vee c_y \vee c_x = y \vee c_x \vee c_y$$

and  $[x] = [y]$  by the definition (3.13). □

Furthermore, Proposition 3.11 leads us to the following.

**Proposition 3.26.** *Let  $\{M_\lambda\}_{\lambda \in \Lambda}$  be a family of monoidally distributive posets. Then, the following are equivalent.*

- (1) GRC holds on  $M_\lambda$  for any  $\lambda \in \Lambda$ .
- (2) GRC holds on  $\prod M_\lambda$ .

Here, GRC is GRC1 or GRC2.

As an application, we extend a result of Dwyer and Palmieri:

**Theorem 3.27** (Dwyer-Palmieri [3]). *There is a ring  $\Lambda$  such that a derived category  $D(\Lambda)$  on  $\Lambda$  does not hold the retract conjecture.*

In the proof of it, Dwyer and Palmieri define  $\Lambda$  to be a truncated polynomial ring over a field  $k$ , and take  $\langle k \rangle$  instead of  $h = \langle H\mathbb{Z}/p \rangle$ . Here  $\langle k \rangle$  denotes a Bousfield class of a complex  $\{X_i\}$  with  $X_0 = k$ , and  $X_i = 0$  if  $i \neq 0$ . By a similar argument of Hovey and Palmieri in [5], if  $r_*$  is an isomorphism from  $\mathbb{B}(D(\Lambda))/J(\langle k \rangle)$  to  $DL$ , then any Bousfield class  $x \in \mathbb{B}(D(\Lambda))$  satisfies  $x^2 = x^3$ . They show the theorem by constructing a Bousfield class  $y \in \mathbb{B}(D(\Lambda))$  such that  $y > y^2 > \cdots > y^n > \cdots$ . By Theorem 3.16, the existence of the class  $y$  implies further the following:

**Theorem 3.28.** *The map  $r_*: \mathbb{B}(D(\Lambda))/N \rightarrow DL$  is not isomorphic.*

#### 4. A BOUSFIELD LATTICE ASSOCIATED TO A QUOTIENT OF PID

We abbreviate ‘principal ideal domain’ to ‘PID’. Furthermore, we write  $x$  for  $\langle x \rangle \in \beta(M)$ , where no confusion arises.

**Theorem 4.1.** *Let  $P$  be a PID and put  $q = p_0^{e_0} \cdots p_{m-1}^{e_{m-1}} \in P$  for prime elements  $p_i$  and integers  $e_i > 0$ . Let  $B$  denote a Bousfield lattice  $\beta(P/qP)$ . Then,*

- (1)  $B = \{x \in P: x \mid q\}$  as a set. In particular  $q$  is the zero element 0.
- (2)  $x \geq y$  if and only if  $x \mid y$ .
- (3)  $DL = \{p_0^{s_0} \cdots p_{m-1}^{s_{m-1}}: s_i = 0 \text{ or } e_i\}$ .
- (4)  $N = \{x \in B: p_0 \cdots p_{m-1} \mid x \text{ in } P\}$ .
- (5)  $B = \prod_{i=0}^{n-1} \beta(P/p_i^{e_i}P)$ .

*Proof.* For an element  $x \in P$ , we consider an integer  $e_i(x)$  and an element  $x_{(q)}$  defined by

$$e_i(x) := \max\{e: e \leq e_i \text{ and } p_i^e \mid x\}, \quad \text{and} \quad x_{(q)} := \prod_{0 \leq i < m} p_i^{e_i(x)}.$$

We see that

$$(4.2) \quad x = x_{(q)} \in \beta(P/qP) \quad \text{for any } x \in P.$$

Indeed,  $x_{(q)}$  divides  $x$ , and so  $x \leq x_{(q)}$ . If  $xy = 0$  in  $P/qP$ , then  $xy$  is divided by  $q$  in  $P$ . Therefore,  $q \mid x_{(q)}y_{(q)}$  and so  $q \mid x_{(q)}y$ . Hence  $x_{(q)}y = 0$  in  $P/qP$  and so  $x_{(q)} \leq x$ .

The statements (1)-(4) follow immediately from (4.2), and (5) from (1).  $\square$

**Corollary 4.3.** *We have isomorphisms of monoidal posets*

$$\begin{aligned} \beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}}P) &= \prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z}) \quad \text{and} \\ DL(\beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}}P)) &= \prod_{i=0}^{n-1} \mathbb{Z}/2. \end{aligned}$$

**Corollary 4.4.** *For any PID  $P$  and a non-zero element  $q \in P$ , the Bousfield lattice  $\beta(P/qP)$  is monoidally distributive.*

*Proof.* Noticing the relation

$$(p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee (p_0^{t_0} \cdots p_{n-1}^{t_{n-1}}) = p_0^{\ell_0} \cdots p_{n-1}^{\ell_{n-1}} \quad \text{with } \ell_i = \min\{s_i, t_i\},$$

the proof is straightforward.  $\square$

**Theorem 4.5.** *If  $P$  is a PID and  $q \in P \setminus \{0\}$ , then GRC2 holds on  $\beta(P/qP)$ , and so does GRC1.*

*Proof.* The ideal  $N(\beta(P/qP))$  has the greatest element  $g = p_0 \cdots p_{n-1}$ . We compute

$$\begin{aligned} (p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee g &= p_0^{\min\{s_0, 1\}} \cdots p_{n-1}^{\min\{s_{n-1}, 1\}} = p_0^{\min\{2s_0, 1\}} \cdots p_{n-1}^{\min\{2s_{n-1}, 1\}} \\ &= (p_0^{2s_0} \cdots p_{n-1}^{2s_{n-1}}) \vee g = (p_0^{s_0} \cdots p_{n-1}^{s_{n-1}})^2 \vee g. \end{aligned}$$

So the theorem follows from Theorem 3.23.  $\square$

*Remark 4.6.* We have another proof of the theorem. Since  $\beta(P/qP) = \prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$  and GRC2 holds on  $\beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$ , GRC2 holds on  $\beta(P/qP)$  by Proposition 3.26.

## 5. BOUSFIELD LATTICES OF STABLE HOMOTOPY CATEGORIES

Let  $\mathcal{L}_E$  for a spectrum  $E$  denote the stable homotopy category of  $E$ -local spectra, and  $\mathbb{B}(\mathcal{L}_E)$  the Bousfield lattice in the sense of Bousfield. Then we have the Bousfield localization functor  $L_E: \mathcal{S} \rightarrow \mathcal{L}_E$ . The monoidal structure of  $\mathcal{L}_E$  is given by  $XY = L_E(X \wedge Y)$ . We consider the Johnson-Wilson spectra  $E(n)$  and the Morava  $K$ -theories  $K(n)$  for  $n \geq 0$ . By the chromatic viewpoint, investigating the categories  $\mathcal{L}_n (= \mathcal{L}_{E(n)})$  and  $\mathcal{L}_{K(n)}$  is one of main targets of stable homotopy theory. We determine the Bousfield lattices of these categories.

We begin with a simple category. A spectrum  $F$  is called a *field* if it is a ring spectrum and  $F \wedge X = \bigvee \Sigma^a F$  for all spectra  $X$ .

**Proposition 5.1.** *Let  $F$  be a field. Then,  $\mathbb{B}(\mathcal{L}_F) = \mathbb{Z}/2$ .*

*Proof.* Since  $F$  is a ring spectrum, we have  $FX = F \wedge X$ . We see easily  $\langle X \rangle \geq \langle FX \rangle$ . Suppose that  $(FX)C = 0$ . Then,  $XC$  is  $F$ -acyclic and so  $XC = 0$ . It follows that  $\langle X \rangle = \langle FX \rangle = \langle \bigvee \Sigma^i F \rangle = 0$  or  $\langle F \rangle$ , which shows the lemma.  $\square$

By [4], the Eilenberg-MacLane spectrum  $H\mathbb{Z}/p$  and the Morava  $K$ -theories  $K(n)$  are fields.

**Corollary 5.2.**  $\mathbb{B}(\mathcal{L}_{H\mathbb{Z}/p}) = \mathbb{Z}/2 = \mathbb{B}(\mathcal{L}_{K(n)})$ .

**Theorem 5.3.** *Let  $p_0, \dots, p_n$  be  $n+1$  distinguished primes. Then  $\mathbb{B}(\mathcal{L}_n)$  is isomorphic to  $\beta(\mathbb{Z}/p_0 \cdots p_n) = \prod_{i=0}^n \mathbb{Z}/2$  in  $\mathcal{MP}$ .*

*Proof.* The Bousfield lattice  $\mathbb{B}(\mathcal{L}_n)$  consists of  $\langle L_n X \rangle$  for all spectra  $X$ , which equals, by Ravenel [9],

$$\begin{aligned} \langle L_n X \rangle &= \langle L_n S^0 \rangle \cdot \langle X \rangle = \langle E(n) \rangle \cdot \langle X \rangle \\ &= \left( \bigvee_{0 \leq i \leq n} \langle K(i) \rangle \right) \cdot \langle X \rangle = \bigvee_{0 \leq i \leq n \text{ and } K(i) \wedge X \neq 0} \langle K(i) \rangle. \end{aligned}$$

since  $L_n$  is smashing and  $K(n)$  is a field. Here  $\langle X \rangle \cdot \langle Y \rangle$  is the Bousfield class of the smash product  $X \wedge Y$ . We define a map  $f: \mathbb{B}(\mathcal{L}_n) \rightarrow \beta(\mathbb{Z}/p_0 \cdots p_n)$  by  $f(\bigvee_{i \in S} \langle K(i) \rangle) = \prod_{i \notin S} p_i$  for  $S \subset \{0, 1, \dots, n\}$ . Then  $f$  preserves multiplication, since

$$\begin{aligned} (\bigvee_{i \in S} \langle K(i) \rangle) (\bigvee_{j \in T} \langle K(j) \rangle) &= \bigvee_{i \in S \cap T} \langle K(i) \rangle, \\ (\prod_{i \notin S} p_i) (\prod_{j \notin T} p_j) &= \prod_{i \notin S \text{ or } i \notin T} p_i = \prod_{i \notin S \cap T} p_i. \end{aligned}$$

Moreover, for the order, we have

$$\begin{aligned} \bigvee_{i \in S} \langle K(i) \rangle \leq \bigvee_{i \in T} \langle K(i) \rangle &\Leftrightarrow S \subset T \Leftrightarrow I(n) - S \supset I(n) - T \\ &\Leftrightarrow \prod_{i \notin S} p_i \leq \prod_{i \notin T} p_i, \end{aligned}$$

and  $f$  is a monoidal poset map.  $\square$

A similar argument shows the following

**Theorem 5.4.** *Let  $E = \bigvee_{i \in F} K(i)$  be a spectrum for a finite subset  $F$  of  $\mathbb{Z}_{\geq 0}$ . Then  $\mathbb{B}(\mathcal{L}_E)$  is isomorphic to  $\prod_{i \in F} \mathbb{Z}/2$ .*

This together with Theorem 4.5 implies

**Corollary 5.5.** *GRC2 holds on  $\mathbb{B}(\mathcal{L}_E)$  for a spectrum  $E = \bigvee_{i \in F} K(i)$  on a finite subset  $F$  of  $\mathbb{Z}_{\geq 0}$ .*

The chromatic tower  $\mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \cdots$  induces the inverse system

$$(5.6) \quad \mathbb{B}(\mathcal{L}_0) \leftarrow \mathbb{B}(\mathcal{L}_1) \leftarrow \mathbb{B}(\mathcal{L}_2) \leftarrow \cdots .$$

Moreover, we notice that  $B_\infty := \lim_n \mathbb{B}(\mathcal{L}_n) = \prod_n \mathbb{Z}/2$  in  $\mathcal{MP}$ . We call a spectrum *harmonic* if it is  $(\bigvee_{i \geq 0} K(i))$ -local.

**Theorem 5.7.** *Let  $\mathcal{H}$  be the stable homotopy category of harmonic spectra. Then  $\mathbb{B}(\mathcal{H})$  is isomorphic to  $B_\infty$  in  $\mathcal{MP}$ .*

*Proof.* Let  $f: \prod \mathbb{Z}/2 \rightarrow \mathbb{B}(\mathcal{H})$  be the poset map defined by  $(x_n) \mapsto \bigvee_{x_n=1} \langle K(n) \rangle$  and let  $p_n: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{L}_n)$  be the poset map defined by  $\langle X \rangle \mapsto \langle X \rangle \cdot \langle E(n) \rangle$ . Then, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{B}(\mathcal{L}_i) & \leftarrow & \mathbb{B}(\mathcal{L}_j) \\ p_i \uparrow & \begin{array}{c} \nearrow p_j \\ \searrow \end{array} & \uparrow \\ \mathbb{B}(\mathcal{H}) & \xleftarrow{f} & \prod \mathbb{Z}/2 \end{array}$$

for any  $i$  and  $j$  with  $i \leq j$ , since

$$\begin{aligned} p_i f((x_n)) &= p_i(\bigvee_{x_n=1} \langle K(n) \rangle) = \bigvee_{x_n=1} \langle K(n) \rangle \cdot \langle E(i) \rangle \\ &= \bigvee_{i \geq n, x_n=1} \langle K(n) \rangle. \end{aligned}$$

Therefore,  $\mathbb{B}(\mathcal{H})$  is the inverse limit of the above system (5.6) by definition.  $\square$

*Proof of Theorem 1.2.* This follows from Theorem 5.7 and Proposition 3.26.  $\square$

In the same way, we obtain

**Theorem 5.8.** *Let  $T$  be a set of field spectra, and put  $\bigvee T = \bigvee_{F \in T} F$ . Then,  $\mathbb{B}(\mathcal{L}_{\bigvee T}) = \prod \mathbb{Z}/2$ .*

## 6. PROBLEMS

We leave some problems in this section.

**Problem 6.1.** *What is a condition on  $X \xrightarrow{f} Y$  in  $\mathcal{M}_0^{epi}$ , under which  $\beta(f)$  is an isomorphism ?*

Suppose that the problem is settled and we find a map from  $\mathbb{B}$  to a commutative monoid  $Y$  such that  $\beta(f)$  is an isomorphism. Then, we may study  $\mathbb{B} = \beta(\mathbb{B})$  by observing  $\beta(Y)$  by virtue of Proposition 2.16, which may let us consider the lattice from a different viewpoint.

**Problem 6.2.** *Let  $M$  be a monoid with 0. Then, is there a ring  $R$  such that  $\beta(M)$  is isomorphic to  $R$  as a monoid ?*

**Example 6.3.** *Let  $p_0, \dots, p_n$  be  $n+1$  distinguished primes. Then  $\beta(\mathbb{Z}/p_0 \dots p_n) = \prod_{i=0}^n \mathbb{Z}/2$  as a monoid by Theorem 5.3.*

If this is possible, we may approach these from the ring theoretic viewpoint.

**Problem 6.4.** *Are  $\mathbb{B}/J(h)$  and  $\mathbb{DL}$  monoidal posets?*

#### REFERENCES

1. A. K. Bousfield, The Boolean algebra of spectra, *Comment. Math. Helv.* **54** (1979), 368–377.
2. W. G. Dwyer and J. H. Palmieri, Ohkawa’s theorem: there is a set of Bousfield classes. *Proc. Amer. Math. Soc.*, **129** (2001), 881–886.
3. W. G. Dwyer and J. H. Palmieri, The bousfield lattice for truncated polynomial algebras, *Homology, Homotopy and Applications*, **10** (2008), 413–436.
4. M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, *Ann. Math.* **148** (1998), 1–49.
5. M. Hovey and J. H. Palmieri, The structure of the Bousfield lattice, *Contemp. Math.* **239** (1999), 175–196.
6. M. Hovey, J. H. Palmieri and N. P. Strickland, Axiomatic stable homotopy theory, *Mem. Am. Math. Soc.* **610** (1997).
7. S. B. Iyengar and H. Krause, The Bousfield lattice of a triangulated category and stratification, *Math. Z.* **273** (2013), 1215–1241.
8. T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, *Hiroshima Math. J.* **19** (1989), 631–639.
9. D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.*, **106** (1984), 351–414.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-8602, JAPAN

*E-mail address:* ryo\_kato\_1128@yahoo.co.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOCHI UNIVERSITY, KOCHI, 780-8520, JAPAN

*E-mail address:* katsumi@kochi-u.ac.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOCHI UNIVERSITY, KOCHI, 780-8520, JAPAN

*E-mail address:* y.tatehara@gmail.com